B.Sc.(Hons) Part I ; 1st PAPER ; TRIGONOMETRY

Summation of series (continued)

1. Find the sum of the series

$$\sin\alpha + n\sin(\alpha+\beta) + \frac{n(n-1)}{\lfloor 2}\sin(\alpha+2\beta) + \cdots \text{ to } (n+1) \text{ terms.}$$

Soln

Let $S = \sin\alpha + n\sin(\alpha+\beta) + \frac{n(n-1)}{\lfloor 2}\sin(\alpha+2\beta) + \cdots$ to $(n+1)$ terms.

Let $C = \cos\alpha + n\cos(\alpha+\beta) + \frac{n(n-1)}{\lfloor 2}\cos(\alpha+2\beta) + \cdots$ to $(n+1)$ terms.

$\therefore C+iS = (\cos\alpha + i\sin\alpha) + n[\cos(\alpha+\beta) + i\sin(\alpha+\beta)] + \frac{n(n-1)}{\lfloor 2}[\cos(\alpha+2\beta) + i\sin(\alpha+2\beta)] + \cdots$ to $(n+1)$ terms

$$\Rightarrow C+iS = e^{i\alpha} + n e^{i(\alpha+\beta)} + \frac{n(n-1)}{\lfloor 2} e^{i(\alpha+2\beta)} + \cdots \text{ to } (n+1) \text{ terms.}$$

$$= e^{i\alpha}\left[1 + n e^{i\beta} + \frac{n(n-1)}{\lfloor 2} e^{2i\beta} + \cdots \text{ to } (n+1) \text{ terms}\right]$$

Put $e^{i\beta} = x$

$$\Rightarrow C+iS = e^{i\alpha}\left[1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \cdots \text{ to } (n+1) \text{ terms}\right]$$

$$\Rightarrow C+iS = e^{i\alpha}(1+x)^n = e^{i\alpha}\left[1 + e^{i\beta}\right]^n$$

$$\Rightarrow C+iS = (\cos\alpha + i\sin\alpha)\left[1+\cos\beta + i\sin\beta\right]^n$$

$$= (\cos\alpha + i\sin\alpha)\left[2\cos^2\frac{\beta}{2} + 2i\sin\frac{\beta}{2}\cos\frac{\beta}{2}\right]^n$$

$$= (\cos\alpha + i\sin\alpha)\left[2\cos\frac{\beta}{2}\left(\cos\frac{\beta}{2} + i\sin\frac{\beta}{2}\right)\right]^n$$

$$\Rightarrow C+iS = (\cos\alpha + i\sin\alpha)\left[2^n\cos^n\frac{\beta}{2}\left(\cos\frac{\beta}{2} + i\sin\frac{\beta}{2}\right)^n\right] \quad -(1)$$

By Demoivre's theorem, ~~we~~ have

$$\left(\cos\frac{\beta}{2} + i\sin\frac{\beta}{2}\right)^n = \cos\frac{n\beta}{2} + i\sin\frac{n\beta}{2}.$$

and

$$(\cos\alpha + i\sin\alpha).\left(\cos\frac{\beta}{2} + i\sin\frac{\beta}{2}\right)^n$$

$$= (\cos\alpha + i\sin\alpha)\left(\cos\frac{n\beta}{2} + i\sin\frac{n\beta}{2}\right)$$

$$= \cos\left(\alpha + \frac{n\beta}{2}\right) + i\sin\left(\alpha + \frac{n\beta}{2}\right) \quad —— (2)$$

So, using (2) in (1), we get

$$C+iS = 2^n\cos^n\frac{\beta}{2}\left[\cos\left(\alpha+\frac{n\beta}{2}\right) + i\sin\left(\alpha+\frac{n\beta}{2}\right)\right]$$

Equating imaginary parts, we get

$$S = 2^n\cos^n\frac{\beta}{2} \cdot \sin\left(\alpha + \frac{n\beta}{2}\right)$$

Q. Sum the series

$$1+\frac{1}{2}\cos\alpha+\frac{1.3}{2.4}\cos 2\alpha+\frac{1.3.5}{2.4.6}\cos 3\alpha+\cdots \text{ to } \infty$$

Soln Let the sum of the series be denoted by C.

$$\therefore C = 1+\frac{1}{2}\cos\alpha+\frac{1.3.}{2.4}\cos 2\alpha+\frac{1.3.5}{2.4.6}\cos 3\alpha+\cdots \text{ to } \infty.$$

Let $S = \frac{1}{2}\sin\alpha+\frac{1.3.}{2.4}\sin 2\alpha+\frac{1.3.5}{2.4.6}\sin 3\alpha+\cdots \text{ to } \infty$

$$\Rightarrow C+iS = 1+\frac{1}{2}(\cos\alpha+i\sin\alpha)+\frac{1.3}{2.4}(\cos 2\alpha+i\sin 2\alpha)$$
$$+\frac{1.3.5}{2.4.6}(\cos 3\alpha+i\sin 3\alpha)+\cdots \text{ to } \infty$$

$$\Rightarrow C+iS = 1+\frac{1}{2}e^{i\alpha}+\frac{1.3}{2.4}e^{2i\alpha}+\frac{1.3.5}{2.4.6}e^{3i\alpha}+\cdots \text{ to } \infty$$

Put $e^{i\alpha} = x$

$$\Rightarrow C+iS = 1+\frac{x}{2}+\frac{1.3.}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\cdots \text{ to } \infty$$

$$\Rightarrow C+iS = (1-x)^{-1/2} = \frac{1}{\sqrt{1-x}}$$

$$\Rightarrow C+iS = \frac{1}{\sqrt{1-e^{i\alpha}}} = \frac{1}{\sqrt{1-\cos\alpha-i\sin\alpha}}$$

$$\Rightarrow C+iS = \frac{1}{\sqrt{2\sin^2\frac{\alpha}{2}-2i\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}}}$$

$$\Rightarrow \quad c + is = \frac{1}{\sqrt{2\sin\frac{\alpha}{2}\cdot\left(\sin\frac{\alpha}{2} - i\cos\frac{\alpha}{2}\right)}}$$

$$\Rightarrow \quad c + is = \frac{1}{\sqrt{2\sin\frac{\alpha}{2}\cdot \cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)}}$$

$$\Rightarrow \quad c + is = \frac{1}{\sqrt{2\sin\frac{\alpha}{2}}\cdot\sqrt{\cos\left(\frac{\pi-\alpha}{2}\right) - i\sin\left(\frac{\pi-\alpha}{2}\right)}}$$

$$\Rightarrow \quad c + is = \frac{\sqrt{2}}{2}\left(\sin\frac{\alpha}{2}\right)^{-1/2}\cdot\left[\cos\left(\frac{\pi-\alpha}{2}\right) - i\sin\left(\frac{\pi-\alpha}{2}\right)\right]^{1/2}$$

$$\Rightarrow \quad c + is = \frac{\sqrt{2}}{2}\left(\sin\frac{\alpha}{2}\right)^{-1/2}\cdot\left[\cos\frac{\pi-\alpha}{4} - i\sin\left(\frac{\pi-\alpha}{4}\right)\right]$$

Equating real parts, we get

$$\Rightarrow \quad c = \frac{\sqrt{2}}{2}\cdot\left(\sin\frac{\alpha}{2}\right)^{-1/2}\cdot\cos\left(\frac{\pi-\alpha}{4}\right)$$